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MATRIC BIALGEBRAS AND QUANTUM GROUPS

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Dedicated to Shimshon Amitsur

ABSTRACT
Generalized determinants and cofactor matrices for matric bialgebras are investigated. Quantum orthogonal and symplectic groups and corresponding quantum general linear groups are introduced as applications.

Introduction

Let M be a bialgebra over a field k. We say M is a *matric bialgebra* if it has a set of generators x_{ij}, $1 \leqq i, j \leqq n$, such that

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}, \qquad \varepsilon(x_{ij}) = \delta_{ij},$$

where Δ and ε denote the comultiplication and the counit. As an easy consequence of the finiteness theorem of coalgebras [S, p. 46] we see every finitely generated bialgebra is a matric bialgebra. It is always possible to construct the Hopf algebra H associated with M, by adding the entries of

$$X^{\mathrm{i}}, X^{\mathrm{iti}}, X^{\mathrm{ititi}}, \ldots$$

where $X = (x_{ij})$, and $(\ )^{\mathrm{i}}$ and $(\ )^{\mathrm{t}}$ denote the inverse and the transpose of the matrix [M2, §7], [T1, §1], [FRT2, §1, Thm. 8]. This general construction is of course very awkward. In case M is commutative, the determinant $g = |X|$ is a group-like element of M, and it is enough to add its inverse to get H, i.e., $H = M[g^{-1}]$.

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A similar situation occurs in quantum group theory. There is a matric bialgebra $M_q(n)$ which is a non-commutative polynomial algebra in t_{ij}, $1 \leqq i, j \leqq n$. Here, q is a non-zero element in k, and the commutation relation among t_{ij} is expressed in terms of q [D, (16)–(19), p. 810]. We review its construction in §2. In this case, the q-determinant $g = |T|_q$, where $T = (t_{ij})$, is a central group-like element in $M_q(n)$, and we have only to add its inverse to get the associated Hopf algebra H, thus $H = M_q(n)[g^{-1}]$. This Hopf algebra represents the so-called *quantum* GL, $\mathrm{GL}_q(n)$ [FRT1, §2], [M1, §1]. In addition, there is an $n \times n$ matrix $\tilde{T}$, the *q-cofactor matrix*, with entries in $M_q(n)$ such that we have $T\tilde{T} = gI = \tilde{T}T$, where I denotes the identity matrix.

Motivated by this example, we investigate matric bialgebras with generalized determinants and cofactor matrices.

Let V be the canonical n-dimensional comodule for the free matric bialgebra $\mathscr{M}_n$. With a subspace W (resp. an endomorphism f) of $V^{(m)}$, the m-fold tensor product of V, we associate a coideal $N(W)$ (resp. $Z(f)$), called the *conormalizer* (resp. *cocentralizer*), of $\mathscr{M}_n$. In §3, we develop generalities on this construction, and give a criterion Theorem 3.8 in order for the relevant quotient bialgebras such as $\mathscr{M}_n/(N(W))$ or $\mathscr{M}_n/(Z(f))$ to have determinants and cofactor matrices. The previously mentioned fact on $M_q(n)$ follows directly from this criterion.

There is an endomorphism τ_q of $V^{(2)}$ such that $M_q(n) = \mathscr{M}_n/(Z(\tau_q))$ [D, p. 817], [FRT1, §2]. We call τ_q the *q-twist map* of type A, since it comes from the constant term of quantum R matrix of type $A^{(1)}$ [J2]. In §4, we introduce two more analogues $\tau_q^{\pm}$, and investigate the associated bialgebras $M_q^{\pm}(n) = \mathscr{M}_n/(Z(\tau_q^{\pm}))$. If $q^2 \neq 1$, the matric bialgebra $M_q^{\pm}(n)$ has a natural group-like element γ of degree 2 as well as the "q-transpose" T', where T denotes the canonical generating matrix, such that $TT' = \gamma I = T'T$. (γ and T' depend on the signature $\pm$.) Hence γ is central, and $A_q^{\pm}(n) = M_q^{\pm}(n)/(\gamma - 1)$ is a Hopf algebra. The Hopf algebra $A_q^{+}(n)$ (resp. $A_q^{-}(n)$) represents a q-analogue of the orthogonal (resp. symplectic) group, which we denote by $\mathrm{O}_q(n)$ (resp. $\mathrm{Sp}_q(n)$). (The same idea appears in [FRT2].)

In the classical case, $q = 1$, the groups $\mathrm{O}(n)$ and $\mathrm{Sp}(n)$ are subgroups of $\mathrm{GL}(n)$. But it seems difficult to embed their q-analogues $\mathrm{O}_q(n)$ and $\mathrm{Sp}_q(n)$ into the previously mentioned $\mathrm{GL}_q(n)$. We consider this is because the types are different. In §5, §6, we try to construct two other q-analogues $\mathrm{GL}_q^{\pm}(n)$ such that $\mathrm{O}_q(n)$ (resp. $\mathrm{Sp}_q(n)$) is embeddable into $\mathrm{GL}_q^{+}(n)$ (resp. $\mathrm{GL}_q^{-}(n)$).

Manin [M1, 2] introduces q-analogues of exterior and symmetric algebras, $\Lambda_q(V)$ and $S_q(V)$. In our terminology, they are defined by the two eigenspaces of τ_q. If $q^2 \neq -1$, $M_q(n)$ can be reformulated by means of $\Lambda_q(V)$ and $S_q(V)$ [M2, §1, Thm. 3], since τ_q is then diagonalizable and the cocentralizer $Z(\tau_q)$ is the sum of the conormalizers of the eigenspaces. In §5, we construct their orthogonal (resp. symplectic) analogues $\Lambda_q^{\pm}(V)$ and $S_q^{\pm}(V)$, and the associated matric bialgebra $\tilde{M}_q^{\pm}(n)$, by means of the eigenspaces of $\tau_q^{\pm}$. We see $M_q^{\pm}(n)$ is a quotient bialgebra of $\tilde{M}_q^{\pm}(n)$, and these new q-analogues of exterior and symmetric algebras have similar bases as the classical case. In particular, the 1-dimensional n-th component of $\Lambda_q^{\pm}(V)$ determines a group-like element, called the *q-determinant of orthogonal* (resp. *symplectic*) *type*, of degree n in $\tilde{M}_q^{\pm}(n)$. We expect there is a cofactor matrix relative to this new q-determinant. However, the problem is not so easy as the case of $M_q(n)$ of type A. The final Section §6 is devoted to this problem. Our results (6.8, 6.9) show that it is enough to evaluate at most $n+1$ elements in k. For small values of n such as 3, 5, we can evaluate these elements, but the general case is open. In the least example $\tilde{M}_q^{+}(3)$, we can write down its defining relations explicitly to see it is a non-commutative polynomial algebra in the entries of the generating matrix. The general case is still open, though we expect it is true.

The results of this paper have been announced in [T2] without proofs.

1. Matric bialgebras and generalizations of the determinant and the cofactor matrix

Throughout the paper we work over a fixed field k. By a *matric bialgebra* of rank n, we mean a bialgebra M which is generated by elements x_{ij}, $1 \leqq i, j \leqq n$, such that we have

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}, \qquad \varepsilon(x_{ij}) = \delta_{ij}.$$

The $n \times n$ matrix $X = (x_{ij})$ is called a *generating matrix* for M (cf. [M2, §2.6]). Every matric bialgebra is a quotient of the *free* (or *universal*) *matric bialgebra*

$$\mathcal{M}_n = k\langle t_{11}, \ldots, t_{ij}, \ldots, t_{nn}\rangle.$$

We are interested in construction of the associated Hopf algebras (cf. [FRT2, §1.6, Thm. 8], [M2, §7]). For a matrix bialgebra M with a generating $n \times n$ matrix X, consider the following conditions:

(1.1) *There are a group-like element g in M and an $n \times n$ matrix $\tilde{X}$ with entries in M such that we have*

$$X\tilde{X} = gI = \tilde{X}X,$$

where I denotes the identity matrix.

(1.2) *If R is an algebra and if $\alpha : M \to R$ is an algebra map such that the matrix $\alpha(X)$ is invertible, there is an anti-morphism $\beta : M \to R$ such that $\beta(X) = \alpha(X)^{-1}$.*

If M is commutative, both conditions are satisfied with $g = |X|$, the determinant, and $\tilde{X}$, the cofactor matrix. So, the g and $\tilde{X}$ in (1.1) are considered as generalizations of the determinant and the cofactor matrix. Note that (1.1) implies g is central, since we have $gX = X\tilde{X}X = Xg$. Generally, for a central group-like element g in a bialgebra M, the localization $M[g^{-1}]$ has an extended bialgebra structure with g^{-1} group-like.

Proposition 1.3. *Conditions* (1.1) *and* (1.2) *imply that $M[g^{-1}]$ is a Hopf algebra, i.e., has an antipode.*

Proof. By (1.2), there is an anti-morphism $S : M \to M[g^{-1}]$ such that $S(X) = g^{-1}\tilde{X}$. We have $i * S = u\varepsilon = S * i$ under the convolution product [S, §4.0], with the canonical map i and the unit u. Hence we have $S(g) = g^{-1}$, and S extends to the antipode. Q.E.D.

A non-commutative example of a matric bialgebra satisfying (1.1) and (1.2) is given in the next section.

2. q-Matrices, the q-determinant, and the q-cofactor matrix

Let q be a non-zero element in k. Let $X = (x_{ij})$ be an $n \times n$ matrix with entries in some algebra. We say X is a *q-matrix* if the following conditions are satisfied (cf. [D, (16)–(19), p. 810]):

(2.1) *If $(a_1, \ldots, a_n)$ is a row or a column of X, we have*

$$a_j a_i = q a_i a_j, \qquad 1 \leqq i < j \leqq n.$$

(2.2) *If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 minor of X, we have*

$$bc = cb, \quad da - ad = (q - q^{-1})bc.$$

For a q-matrix X, the *q-determinant* $|X|_q$ is defined as follows:

$$|X|_q = \sum_{\sigma \in S_n} (-q)^{-l(\sigma)} x_{1\sigma(1)} \cdots x_{n\sigma(n)},$$

where $l(\sigma)$ denotes the number of inversions of σ [D, ibid.], [M1, §3]. Let $X_{j,i}$ be the $(n-1) \times (n-1)$ minor of X obtained by removing the j-th row and the i-th column. The matrix

$$\tilde{X} = ((-q)^{j-i} |X_{j,i}|_q)_{ij}$$

is called the *q-cofactor matrix*. The following *q-Laplace formula* is implicit in [FRT1, Thm. 4] (cf. also [TT, Prop. 2.10]).

PROPOSITION 2.3. *If X is a q-matrix, we have*

$$X\tilde{X} = |X|_q I = \tilde{X}X.$$

Consider conditions (2.1) and (2.2) for the generating matrix $T = (t_{ij})$ of the universal matric bialgebra $\mathcal{M}_n$. These determine an ideal I_q of $\mathcal{M}_n$, which is the least one making T into a q-matrix modulo I_q. We put $M_q(n) = \mathcal{M}_n / I_q$.

PROPOSITON 2.4. (a) *I_q is a bi-ideal of $\mathcal{M}_n$, i.e., we have*

$$\Delta(I_q) \subset I_q \otimes \mathcal{M}_n + \mathcal{M}_n \otimes I_q, \qquad \varepsilon(I_q) = 0.$$

(b) *$|T|_q$ is a group-like element in $M_q(n)$.*
(c) *The matric bialgebra $M_q(n)$ satisfies conditions* (1.1) *and* (1.2) *relative to $g = |T|_q$ and the q-cofactor matrix $\tilde{T}$.*

Proofs of Propositions 2.3 and 2.4 will be given in §4 (cf. [M2, §1, Thm. 3]). The Hopf algebra $M_q(n)[|T|_q^{-1}]$ (resp. $M_q(n)/(|T|_q - 1)$) corresponds with $\mathrm{GL}_q(n)$, quantum GL (resp. $\mathrm{SL}_q(n)$, quantum SL) [D], [M1, 2].

3. Conormalizers and cocentralizers

Fix an integer n, and put $V = k^n$ with canonical base $v_1, \ldots, v_n$ throughout the rest of the paper. We use the natural right $\mathcal{M}_n$-comodule structure

$$\rho(v_j) = \sum_{i=1}^{n} v_i \otimes t_{ij}, \qquad j = 1, \ldots, n,$$

where t_{ij} are the canonical generators for $\mathcal{M}_n$. For any $m \geqq 0$, let $V^{(m)}$ be the m-fold tensor product of V, which is also a right $\mathcal{M}_n$-comodule, since $\mathcal{M}_n$ is a bialgebra. We introduce an inner product $\langle\ ,\ \rangle$ on $V^{(m)}$ such that

$v_{i_1}\otimes\cdots\otimes v_{i_m}$, $1\leqq i_1,\ldots,i_m\leqq n$, form an orthonormal base. We define $\phi: V^{(m)}\otimes V^{(m)}\to \mathcal{M}_n$ by putting

$$\phi(v_{(i)}\otimes v_{(j)})=t_{(i)(j)},$$

where $v_{(i)}=v_{i_1}\otimes\cdots\otimes v_{i_m}$ and $t_{(i)(j)}=t_{i_1j_1}\cdots t_{i_mj_m}$ for $(i)=(i_1,\ldots,i_m)$ and $(j)=(j_1,\ldots,j_m)$. The $[1,n]^m\times[1,n]^m$ matrix $(t_{(i)(j)})$ will be denoted by $T^{(m)}$. We have

$$\phi(x\otimes y)=\sum_\alpha \langle x,y_\alpha\rangle z_\alpha, \tag{3.1}$$

$$\Delta\phi(x\otimes y)=\sum_\alpha \phi(x\otimes y_\alpha)\otimes z_\alpha, \tag{3.2}$$

if $\rho(y)=\Sigma_\alpha y_\alpha\otimes z_\alpha$, for x,y in $V^{(m)}$.

For a subspace W of $V^{(m)}$, we put

$$N(W)=\phi(W^\perp\otimes W),$$

where $W^\perp=\{x\in V^{(m)}\mid\langle x,W\rangle=0\}$.

For an endomorphism f of $V^{(m)}$, we put

$$Z(f)=\operatorname{Im}(\phi\circ(I\otimes f-f^*\otimes I)),$$

where f^* denotes the adjoint of f, i.e., the map such that

$$\langle f^*(x),y\rangle=\langle x,f(y)\rangle,\qquad \text{for } x,y \text{ in } V^{(m)}.$$

PROPOSITION 3.3. (a) *$N(W)$ is a coideal. It is the least subspace such that*

$$\rho(W)\subset W\otimes\mathcal{M}_n+V^{(m)}\otimes N(W).$$

(b) *$Z(f)$ is a coideal. It is the least subspace such that*

$$\operatorname{Im}((f\otimes I)\circ\rho-\rho\circ f)\subset V^{(m)}\otimes Z(f).$$

(c) *Let F be the matrix of f relative to the base $v_{(i)}$, $(i)\in[1,n]^m$. Then $Z(f)$ is spanned by the entries of the $[1,n]^m\times[1,n]^m$ matrix $T^{(m)}F-FT^{(m)}$.*

(d) *If f is diagonalizable with eigenspaces W_λ, we have*

$$Z(f)=\bigoplus_\lambda N(W_\lambda).$$

PROOF. The second assertions of (a), (b) follow directly from (3.1), and the first assertions follow from this, together with (3.2). (c) is a restatement of (b). (d) For any $\alpha\in k$, we have $\operatorname{Im}(f^*-\alpha)=\operatorname{Ker}(f-\alpha)^\perp$. Hence we have

$$(I \otimes f - f^* \otimes I)(V^{(m)} \otimes W_\lambda) = W_\lambda^\perp \otimes W_\lambda.$$

This implies (d). Q.E.D.

$N(W)$ (resp. $Z(f)$) is called the *conormalizer* of W (resp. the *concentralizer* of f). It follows that $N(W)$ (resp. $Z(f)$) is the least subspace, in fact the least coideal, such that W is a subcomodule of V for $\mathcal{M}_n/N(W)$ (resp. f is a comodule endomorphism for $\mathcal{M}_n/Z(f)$).

PROPOSITION 3.4. (a) *For a linear endomorphism f of $V^{(m)}$, the matric bialgebra $\mathcal{M}_n/(Z(f))$, quotient by the bi-ideal generated by $Z(f)$, satisfies condition* (1.2).

(b) *If we have a vector space decomposition $V^{(m)} = W_1 \oplus \cdots \oplus W_r$, the quotient matric bialgebra $\mathcal{M}_n/(N(W_1), \ldots, N(W_r))$ satisfies condition* (1.2).

PROOF. Take an invertible $n \times n$ matrix $A = (a_{ij})$ in an algebra R with $A^{-1} = (b_{ij})$. Define an algebra map $\alpha : \mathcal{M}_n \to R$ and an anti-morphism $\beta : \mathcal{M}_n \to R$ by setting $\alpha(t_{ij}) = a_{ij}$ and $\beta(t_{ij}) = b_{ij}$. Put

$$g = (I \otimes \alpha) \circ \rho \quad \text{and} \quad h = (I \otimes \beta) \circ \rho$$

which are maps $V^{(m)} \to V^{(m)} \otimes R$. Extend them into right R-linear endomorphisms $\bar{g}$, $\bar{h}$ of $V^{(m)} \otimes R$. They are inverses of each other. It follows from Proposition 3.3 that α vanishes on $Z(f)$ (resp. on $N(W_1) + \cdots + N(W_r)$) if and only if $\bar{g}$ commutes with $f \otimes I$ (resp. $\bar{g}$ induces automorphisms of $W_i \otimes R$ for all i). If this is the case, the same holds for its inverse $\bar{h}$. This means β vanishes on the same space yielding the required anti-morphism. Q.E.D.

There is a natural algebra automorphism θ of $\mathcal{M}_n$ such that $\theta(t_{ij}) = t_{ji}$, $1 \leqq i, j \leqq n$. It is a coalgebra anti-morphism, and we have

$$\theta(\phi(x \otimes y)) = \phi(y \otimes x), \qquad \text{for } x, y \text{ in } V^{(m)}.$$

We are interested in θ-stable bialgebra quotients. For instance, if f is a *symmetric*, i.e., such that $f^* = f$, endomorphism of $V^{(m)}$, then $Z(f)$ is θ-stable. If further f is diagonalizable, then the eigenspaces W_λ are orthogonal to each other, and we have

$$Z(f) = \bigoplus_{\lambda \neq \mu} \phi(W_\lambda \otimes W_\mu).$$

LEMMA 3.5. *Let L be a θ-stable coideal of $\mathcal{M}_n$. If W is a subspace of $V^{(m)}$ such that*

$$\rho(W) \subset W \otimes \mathcal{M}_n + V^{(m)} \otimes L$$

then $W^{\perp} \subset V^{(m)}$ *also satisfies*

$$\rho(W^{\perp}) \subset W^{\perp} \otimes \mathcal{M}_n + V^{(m)} \otimes L.$$

PROOF. The first inclusion is equivalent with $N(W) \subset L$. This implies $N(W^{\perp}) \subset \theta(L) = L$, the second inclusion. Q.E.D.

We give a criterion in order for a θ-stable bialgebra quotient of $\mathcal{M}_n$ to satisfy condition (1.1).

ASSUMPTION 3.6. We are given the following data: an integer $m > 0$, a symmetric endomorphism f of rank 1 of $V^{(m)}$ with $\mathrm{Im}(f) = kz$, a symmetric endomorphism f' of rank n of $V^{(m-1)}$ with a base $u_1, \ldots, u_n$ for $V^{(m-1)}$ mod. $\mathrm{Ker}(f')$ such that $\langle u_i, f'(u_j)\rangle = \delta_{ij}$. There are a_{ij}, b_{ij}, and $c \neq 0$ in k such that

$$z = \sum_{i,j} a_{ij} v_i \otimes f'(u_j) = \sum_{i,j} b_{ij} f'(u_i) \otimes v_j, \tag{3.7}$$

and $(a_{ij})(b_{ij}) = cI$.

With the assumption, let L be a θ-stable bi-ideal of $\mathcal{M}_n$ containing both $Z(f)$ and $Z(f')$, and let $M = \mathcal{M}_n/L$. By assumption, f and f' are M-comodule endomorphisms. In particular, the 1-dimensional M-comodule kz determines a group-like element g in M with $\rho(z) = z \otimes g$. There are p_{ij} in M such that

$$\rho(f'(u_j)) = \sum_i f'(u_i) \otimes p_{ij}, \qquad 1 \leqq j \leqq n,$$

since $f'(u_i)$ span an M-subcomodule. We see p_{ij} is the image of $\phi(u_i \otimes f'(u_j))$, since $\langle u_i, f'(u_j)\rangle = \delta_{ij}$. It follows that $\theta(p_{ij}) = p_{ji}$, since f' is symmetric and $\theta\phi(u_i \otimes f'(u_j)) = \phi(f'(u_j) \otimes u_i) \equiv \phi(u_j \otimes f'(u_i))$ modulo L.

THEOREM 3.8. *With assumption* 3.6, *let* L *be a* θ-*stable bi-ideal of* $\mathcal{M}_n$ *containing both* $Z(f)$ *and* $Z(f')$, *and let* $M = \mathcal{M}_n/L$. *Put*

$$\tilde{T} = c^{-1}(a_{ij})P^{t}(b_{ij}),$$

where $P = (p_{ij})$ *with* p_{ij} *the image of* $\phi(u_i \otimes f'(u_j))$. *We have in* M

$$T\tilde{T} = gI = \tilde{T}T.$$

PROOF. If we write $z = f(\zeta)$ with $\zeta \in V^{(m)}$, then $\langle z, \zeta\rangle \neq 0$, since $\mathrm{Ker}(f) = \mathrm{Im}(f)^{\perp}$. This implies $\theta(g) = g$, since $\langle z, \zeta\rangle g$ is the image of $\phi(\zeta, f(\zeta))$ which is θ-invariant. Applying ρ to (3.7), we get

(3.9) $$T(a_{ij})P^{t} = g(a_{ij}),$$

(3.10) $$P(b_{ij})T^{t} = g(b_{ij}).$$

Applying θ to (3.10), we get

(3.11) $$P^{t}(b_{ij})T = g(b_{ij}).$$

The assertion follows from (3.9) and (3.11), if we note $(a_{ij})(b_{ij}) = cI$. Q.E.D.

If in particular L is of the form in Proposition 3.4, for instance $L = (Z(f), Z(f'))$, then $M = \mathcal{M}_n/L$ satisfies conditions (1.1) and (1.2), and we get the associated Hopf algebra $M[g^{-1}]$. As an application we deduce Propositions 2.3 and 2.4 in §4.

4. q-Twist maps

We define a symmetric endomorphism τ_q of $V^{(2)}$, called the *q-twist map* of type A, as follows:

$$\tau_q = q\sum_{i} e_{ii}\otimes e_{ii} + \sum_{i\neq j} e_{ij}\otimes e_{ji} + (q-q^{-1})\sum_{i<j} e_{jj}\otimes e_{ii},$$

where e_{ij} denote matrix units [D, p. 817], [J1, p. 250], [FRT2, (1.5)]. When $q = 1$, it is the usual twist map $x\otimes y \mapsto y\otimes x$. τ_q satisfies the braid condition as well as

(4.1) $$(\tau_q - q)(\tau_q + q^{-1}) = 0$$

[J1, ibid.]. Hence it is diagonalizable with eigenvalues $q, -q^{-1}$, if $q^2 \neq -1$.

It is known [D, ibid.] and easily verified that an $n\times n$ matrix X with entries in some algebra is a q-matrix if and only if $X^{(2)}$ commutes with τ_q. It follows that $I_q = (Z(\tau_q))$, hence I_q is a bi-ideal of $\mathcal{M}_n$ and $M_q(n) = \mathcal{M}_n/I_q$ satisfies condition (1.2), by Proposition 3.4.

We prove the other parts of Propositions 2.3 and 2.4 as an application of Theorem 3.8. Take a permutation $\sigma\in S_m$ with $l = l(\sigma)$, the number of inversions. Write σ as a product

(4.2) $$\sigma = (a_1, a_1+1)\cdots(a_l, a_l+1)$$

of transpositions with $1\leqq a_i < m$. For $1\leqq a < m$, put

$$\tau_a = I\otimes\cdots\otimes \underset{(a,\,a+1)}{\tau_q} \otimes\cdots\otimes I \in \mathrm{End}(V^{(m)}).$$

One sees the product

$$\tau_\sigma = \tau_{a_1} \cdots \tau_{a_l}$$

does not depend on the expression (4.2) [J1]. Put

$$f_m = \sum_{\sigma \in S_m} (-q)^{-l(\sigma)} \tau_\sigma \in \mathrm{End}(V^{(m)})$$

("symmetrizer" [ibid.]). We see the pair (f_n, f_{n-1}) as (f, f') satisfies Assumption 3.6 together with the following data:

$$u_i = v_1 \otimes \cdots \otimes v_{i-1} \otimes v_{i+1} \otimes \cdots \otimes v_n, \qquad 1 \leqq i \leqq n,$$

$$\begin{aligned} z &= \sum_{\sigma \in S_n} (-q)^{-l(\sigma)} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \\ &= \sum_i (-q)^{1-i} v_i \otimes f_{n-1}(u_i) = \sum_i (-q)^{i-n} f_{n-1}(u_i) \otimes v_i. \end{aligned}$$

Hence Propositions 2.3 and 2.4 follow from Theorem 3.8.

Next we define the q-twist maps of types B–D, which come from the constant terms of quantum R matrices [J2]. We assign signature $+$ (resp. $-$) to the orthogonal case, i.e., types B, D (resp. the symplectic case, i.e., type C). Sometimes we omit the signature, for instance in T' or γ.

Assume the fixed integer n is even in the symplectic case. Let $q \neq 0$ in k. Assume q has a square root $q^{1/2}$ in k if n is odd. For $1 \leqq i \leqq n$, let $i' = n + 1 - i$. We put

$$\bar{i} = \begin{cases} i - n/2, & i < i' \\ 0, & i = i' \\ i - n/2 - 1, & i > i' \end{cases} \quad \text{and} \quad \bar{i} = \begin{cases} i - n/2 - 1, & i < i' \\ i - n/2, & i > i' \end{cases}$$

(orthogonal case) (symplectic case)

We put $\varepsilon_i = -1$ if $i > i'$ in the symplectic case, and $\varepsilon_i = 1$ for all other cases. The q-twist maps τ_q^+ and τ_q^- are defined as follows:

$$\begin{aligned} \tau_q^{\pm} &= \sum_{i \neq i'} (q e_{ii} \otimes e_{ii} + q^{-1} e_{ii'} \otimes e_{i'i}) \\ &\quad + \sum_{i \neq j, j'} e_{ij} \otimes e_{ji} + e_{n_0 n_0} \otimes e_{n_0 n_0} \\ &\quad + (q - q^{-1}) \sum_{i < j} (e_{jj} \otimes e_{ii} - \varepsilon_i \varepsilon_j q^{\bar{i} - \bar{j}} e_{i'j} \otimes e_{ij'}), \end{aligned} \tag{4.3}$$

where $n_0 = (n+1)/2$, and the term $e_{n_0n_0} \otimes e_{n_0n_0}$ is used only if n is odd (cf. [FRT2, 1.4]). They are symmetric satisfying the braid condition as well as

$$\begin{aligned}(4.4)\qquad &(\tau_q^+ - q)(\tau_q^+ + q^{-1})(\tau_q^+ - q^{1-n}) = 0,\\ &(\tau_q^- - q)(\tau_q^- + q^{-1})(\tau_q^- + q^{-1-n}) = 0.\end{aligned}$$

For a $n \times n$ matrix $X = (x_{ij})$ with entries in an algebra, we put

$$X' = (\varepsilon_i \varepsilon_j q^{\bar{j}-\bar{i}} x_{j'i'})_{ij}.$$

(Note that the concept depends on the signature.)

DEFINITION 4.5. X is called a *q-orthogonal* (resp. *q-symplectic*) matrix if $X^{(2)}$ commutes with τ_q^+ (resp. τ_q^-) and if $XX' = I = X'X$.

DEFINITION 4.6 [T2, Def. 1], [FRT2, Def. 11]. We define quotient bialgebras

$$M_q^{\pm}(n) = \mathcal{M}_n/(Z(\tau_q^{\pm})),$$

$$A_q^{\pm}(n) = M_q^{\pm}(n)/(TT' = I = T'T),$$

where T denotes the canonical generating matrix.

Thus, $A_q^{\pm}(n)$ is the matric bialgebra of rank n generated by a universal q-orthogonal (resp. q-symplectic) matrix.

PROPOSITION 4.7 [T2, Prop. 2], [FRT2, Thm. 5].

(a) *$A_q^{\pm}(n)$ is a Hopf algebra.*

(b) *If $q^2 \neq 1$, there is a group-like element γ (of degree 2) in $M_q^{\pm}(n)$ such that $TT' = \gamma I = T'T$. Thus $M_q^{\pm}(n)$ satisfies conditions* (1.1) *and* (1.2), *and we can identify*

$$A_q^{\pm}(n) = M_q^{\pm}(n)/(\gamma - 1).$$

PROOF. We put

$$e_q^{\pm} = \sum_{i,j} \varepsilon_i \varepsilon_j q^{-\bar{i}-\bar{j}} e_{ij} \otimes e_{i'j'}$$

which is symmetric of rank 1. Let X be an $n \times n$ matrix in an algebra. By a direct calculation, one sees $X^{(2)}$ commutes with $e_q^{\pm}$ if and only if $XX' = X'X = \gamma I$ for some element γ in the algebra. We have $X^{(2)}e_q^{\pm} = \gamma I$ if this is the case. It follows that the bialgebra quotient $\mathcal{M}_n/(Z(e_q^{\pm}))$ satisfies conditions (1.1) and (1.2), relative to (γ, T'), and in particular its quotient by $(\gamma - 1)$ is a Hopf algebra. On the other hand, we have

$$(\tau_q^+ - q)(\tau_q^+ + q^{-1}) = (q^{-1} - q)q^{1-n}e_q^+,$$

$$(\tau_q^- - q)(\tau_q^- + q^{-1}) = (q^{-1} - q)q^{-1-n}e_q^-.$$

(There are further relations for $\tau_q^{\pm}$, $e_q^{\pm}$ leading to the Birman–Wenzl–Murakami algebra.) Since we can identify

$$A_q^{\pm}(n) = (\mathcal{M}_n/(Z(\tau_q^{\pm}), Z(e_q^{\pm}))/(\gamma - 1),$$

(a) follows from the above argument. If $q^2 \neq 1$, $e_q^{\pm}$ is a polynomial in $\tau_q^{\pm}$, hence $Z(e_q^{\pm})$ is contained in $Z(\tau_q^{\pm})$. This yields (b). Q.E.D.

Note that the pair $(e_q^{\pm}, I)$ as (f, f') $(m = 2)$ satisfies Assumption 3.6 together with $u_i = v_i$ and $z = \Sigma_i \varepsilon_i q^{-\bar{i}} v_i \otimes v_{i'}$. The corresponding $\tilde{T}$ is precisely T'.

We denote by $\mathrm{O}_q(n)$ (resp. $\mathrm{Sp}_q(n)$) the quantum group corresponding to the Hopf algebra $A_q^+(n)$ (resp. $A_q^-(n)$), and call it the *quantum orthogonal* (resp. *symplectic*) *group*. (In [FRT2, Def. 11], the same quantum groups are introduced under different symbols.)

5. q-Exterior and q-symmetric algebras

Manin [M1, 2] defines some q-analogues of the exterior and symmetric algebras to reformulate the matric bialgebra $M_q(n)$ of type A. We explain this approach from the viewpoint of §3, and generalize to other types B–D.

Case A. The q-twist τ_q has eigenvalues q, $-q^{-1}$ (4.1). If $q^2 \neq -1$, we have a decomposition into eigenspaces

$$V^{(2)} = W_q \oplus W_{-q^{-1}},$$

yielding $Z(\tau_q) = N(W_q) \oplus N(W_{-q^{-1}}) = \phi(W_q \otimes W_{-q^{-1}}) \oplus \phi(W_{-q^{-1}} \otimes W_q)$ (see above Lemma 3.5). We define the following quotient algebras of the tensor algebra $T(V)$ (quadratic algebras as Manin calls)

$$\Lambda_q(V) = T(V)/(W_q),$$

$$S_q(V) = T(V)/(W_{-q^{-1}}),$$

where (W_q), $(W_{-q^{-1}})$ denote the ideals generated by W_q, $W_{-q^{-1}}$. By means of the canonical base $v_1, \ldots, v_n$ we can rewrite

$$\Lambda_q(V) = k\langle v_1, \ldots, v_n\rangle/(v_i^2, v_i v_j + q v_j v_i \quad (i < j)),$$

$$S_q(V) = k\langle v_1, \ldots, v_n\rangle/(v_j v_i - q v_i v_j \quad (i < j)).$$

The products $v_{i_1}\cdots v_{i_r}$ with $i_1<\cdots<i_r$ (resp. $i_1\leqq\cdots\leqq i_r$) form a base for $\Lambda_q(V)$ (resp. $S_q(V)$). They are called the *q-exterior* and *q-symmetric* algebras of type A (cf. [M1, 1.2, 3.2], [TT, §2]). It follows that $M_q(n)$ can be reformulated as the largest bialgebra quotient of $\mathcal{M}_n$ which co-normalizes both $\Lambda_q(V)$ and $S_q(V)$, if $q^2\neq-1$ (see [M1, 1.4]). In addition, the q-determinant $|T|_q$ is determined by the n-th component of $\Lambda_q(V)$:

$$\rho(v_1\cdots v_n)=v_1\cdots v_n\otimes|T|_q.$$

Case B,D. The q-twist τ_q^+ has eigenvalues q, $-q^{-1}$, q^{1-n} (4.4). *Assume these are distinct.* We have a decomposition into eigenspaces

$$V^{(2)}=W_q^+\oplus W_{q^{1-n}}^+\oplus W_{-q^{-1}}^+,$$

of dimensions $n(n+1)/2-1$, 1, $n(n+1)/2$ respectively. ($W_{q^{1-n}}^+$ is spanned by $z=\Sigma_i q^{-\bar{i}}v_i\otimes v_{i'}$). This yields

$$\begin{aligned}Z(\tau_q^+)&=N(W_q^+)\oplus N(W_{q^{1-n}}^+)\oplus N(W_{-q^{-1}}^+),\\&=\bigoplus_{\lambda\neq\mu}\phi(W_\lambda^+\otimes W_\mu^+)\qquad(\lambda,\mu\text{ running over }q,q^{1-n},-q^{-1}),\end{aligned}$$

and in particular the eigenspaces are subcomodules for $M_q^+(n)$. The group-like element γ of Proposition 4.7(b) corresponds to the 1-dimensional one $W_{q^{1-n}}^+:\rho(z)=z\otimes\gamma$.

By comparison of the dimensions of eigenspaces, we define the following algebras [T2, Def. 3]:

$$\Lambda_q^+(V)=T(V)/(W_q^+,W_{q^{1-n}}^+),\qquad S_q^+(V)=T(V)/(W_{-q^{-1}}^+).$$

(Note that $(W_q^+\oplus W_{q^{1-n}}^+)^\perp=W_{-q^{-1}}^+$).

Proposition 5.1 [T2, Prop. 4]. (a) *The algebra $\Lambda_q^+(V)$ is defined by n generators $v_1,\ldots,v_n$ and the following relations*:

(i) $v_i^2=0$, *if* $i\neq i'$,
(ii) $v_jv_i=-q^{-1}v_iv_j$, *if* $i<j$, $i\neq j'$,
(iii) $v_{i'}v_i=-v_iv_{i'}+(q^{-1}-q)\Sigma_{k<i}q^{i-k-1}v_kv_{k'}$, *if* $i<i'$,
(iv) $v_{n_0}^2=(q^{-1/2}-q^{1/2})\Sigma_{k<n_0}q^{n_0-k-1}v_kv_{k'}$,

where $n_0=(n+1)/2$, *and* (iv) *is required only when n is odd.*

(b) *The products $v_{i_1}\cdots v_{i_r}$ with $i_1<\cdots<i_r$ form a base for $\Lambda_q^+(V)$.*

Proposition 5.2 [T2, Prop. 5]. (a) *The algebra $S_q^+(V)$ is defined by n generators $v_1,\ldots,v_n$ and the following relations*:

(i) $v_j v_i = q v_i v_j$, *if* $i < j$, $i \neq j'$,

(ii) $v_{i'} v_i = v_i v_{i'} + (q^{-1} - q) \sum_{i<k<n_0} q^{i+1-k} v_k v_{k'}$
$+ q^{i+1-n_0}(q^{-1/2} - q^{1/2}) v_{n_0}^2$, *if* $i < i'$,

where $n_0 = (n+1)/2$, *and the last term in* (ii) *is required only when n is odd.*

(b) *The products* $v_{i_1} \cdots v_{i_r}$ *with* $i_1 \leqq \cdots \leqq i_r$ *form a base for* $S_q^+(V)$.

PROOF. In both propositions, (b) follows from (a) by an easy application of the Diamond Lemma [B]. To prove (a), it is enough to find generators of the eigenspaces. One sees, W_q^+ is spanned by $v_i \otimes v_i$ $(i \neq i')$, $v_i \otimes v_j + q v_j \otimes v_i$ $(i < j,\ i \neq j')$, $q^{-1} v_i \otimes v_{i'} - v_{i+1} \otimes v_{i'-1} - v_{i'-1} \otimes v_{i+1} + q v_{i'} \otimes v_i$ $(i+1 < n_0)$, and $q^{-1} v_{n_0-1} \otimes v_{n_0+1} - (q^{1/2} + q^{-1/2}) v_{n_0} \otimes v_{n_0} + q v_{n_0+1} \otimes v_{n_0-1}$ (if n is odd), and $W^+_{-q^{-1}}$ by $v_j \otimes v_i - q v_i \otimes v_j$ $(i < j,\ i \neq j')$, $v_i \otimes v_{i'} - q v_{i+1} \otimes v_{i'-1} + q^{-1} v_{i'-1} \otimes v_{i+1} - v_{i'} \otimes v_i$ $(i+1 < n_0)$, and $v_{n_0-1} \otimes v_{n_0+1} + (q^{-1/2} - q^{1/2}) v_{n_0} \otimes v_{n_0} - v_{n_0+1} \otimes v_{n_0-1}$ (if n is odd). The description of (a) follows from this. Q.E.D.

We call $\Lambda_q^+(V)$ and $S_q^+(V)$ the *q-exterior* and *q-symmetric algebras of orthogonal type*. (Cf. [FRT2, Def. 12].)

DEFINITION 5.3.

$$\begin{aligned}\tilde{M}_q^+(n) &= \mathcal{M}_n/(N(W_q^+ \oplus W^+_{q^{1-n}}), N(W^+_{-q^{-1}})) \\ &= \mathcal{M}_n/(\phi((W_q^+ \oplus W^+_{q^{1-n}}) \otimes W^+_{-q^{-1}}), \phi(W^+_{-q^{-1}} \otimes (W_q^+ \oplus W^+_{q^{1-n}})).\end{aligned}$$

This is the largest bialgebra quotient which conormalizes both $\Lambda_q^+(V)$ and $S_q^+(V)$. Obviously, $M_q^+(n)$ is a quotient of it. $\tilde{M}_q^+(n)$ satisfies condition (1.2) by Proposition 3.4(b).

PROPOSITION 5.4 [T2, Prop. 7]. *We have*

$$A_q^+(n) = \tilde{M}_q^+(n)/(TT' = I = T'T).$$

PROOF. The relation $TT' = I = T'T$ implies $T^{(2)}$ commutes with e_q^+ (4.7), and e_q^+ has eigenspaces $W^+_{q^{1-n}}$, $W_q^+ \oplus W^+_{-q^{-1}}$. Hence the quotient in the right-hand side conormalizes each W_q^+, $W^+_{q^{1-n}}$, $W^+_{-q^{-1}}$. Therefore it is a quotient of $M_q^+(n)$. Q.E.D.

By Proposition 5.1(b), there is a 1-dimensional subcomodule $kv_1 \cdots v_n$ of $\Lambda_q^+(V)$ for $\tilde{M}_q^+(n)$. This determines a group-like element g (of degree n) of $\tilde{M}_q^+(n)$: $\rho(v_1 \cdots v_n) = v_1 \cdots v_n \otimes g$. The group-like element g, denoted by $|T|_q$ if there is no confusion with the previous q-determinant, is called the *q-determinant of orthogonal type*.

EXAMPLE 5.5. If $n = 3$,

$$|T|_q = t_{11}t_{22}t_{33} - t_{11}t_{32}t_{23} - t_{31}t_{22}t_{13} + q^{-1}(-t_{12}t_{21}t_{33} + t_{12}t_{31}t_{23} + t_{32}t_{21}t_{13}).$$

This follows from Example 6.11(a).

It is very likely that the bialgebra $\tilde{M}_q^+(n)$ satisfies the following properties just as $M_q(n)$.

PROBLEM 5.6. (a) Is there a cofactor matrix $\tilde{T}$ in $\tilde{M}_q^+(n)$ relative to the q-determinant $|T|_q$ of orthogonal type?

(b) Do the ordered products of t_{ij} (relative to some ordering) form a base for $\tilde{M}_q^+(n)$?

(c) Is $\tilde{M}_q^+(n)$ an integral domain?

We consider Problem (a) in §6. To present $|T|_q$ explicitly is also a question. If all items are affirmative, we may think the Hopf algebra $M_q^+(n)[g^{-1}]$ represents a new q-analogue of $\mathrm{GL}(n)$; denoted by $\mathrm{GL}_q^+(n)$ (which was denoted by $\mathrm{GL}_q^0(n)$ in [T2]), and called the *quantum* GL *of orthogonal type*. Proposition 5.4 means the quantum orthogonal group $\mathrm{O}_q(n)$ is the closed subgroup of $\mathrm{GL}_q^+(n)$ defined by the equation $TT' = I = T'T$ (just as in the classical case). The quantum SL of orthogonal type, $\mathrm{SL}_q^+(n)$, is the subgroup defined by $|T|_q = 1$. As the intersection

$$\mathrm{SO}_q(n) = \mathrm{O}_q(n) \cap \mathrm{SL}_q^+(n)$$

We can well-define the q-analogue of $\mathrm{SO}(n)$.

Case C. This goes quite parallel to the above. Assume the q-twist τ_q^- has *distinct* eigenvalues q, $-q^{-1}$, $-q^{-1-n}$. Let

$$V^{(2)} = W_q^- \oplus W_{-q^{-1-n}}^- \oplus W_{-q^{-1}}^-$$

be the decomposition into eigenspaces. The dimensions are $n(n+1)/2$, 1 (spanned by $z^- = \Sigma_i \varepsilon_i q^{-f} v_i \otimes v_{i'}$), and $n(n+1)/2 - 1$ respectively. By comparison of the dimensions, we put

$$\Lambda_q^-(V) = T(V)/(W_q^-),$$

$$S_q^-(V) = T(V)/(W_{-q^{-1-n}}^-, W_{-q^{-1}}^-).$$

The algebras are called the *q-exterior* and *q-symmetric algebras of symplectic type*. We have similar expressions and bases as Propositions 5.1 and 5.2. We define the bialgebra $\tilde{M}_q^-(n)$ as the largest quotient which conormalizes both $\Lambda_q^-(V)$ and $S_q^-(V)$, just as 5.3. Then the Hopf algebra $A_q^-(n)$ coincides with the quotient of $\tilde{M}_q^-(n)$ by $TT' = I = T'T$, and we can consider problems analogous to 5.6. If we can answer affirmatively, we will have $\mathrm{GL}_q^-(n)$ (denoted by $\mathrm{GL}_q^S(n)$ in [T2]), the *quantum* GL *of symplectic type*. It is likely $\mathrm{Sp}_q(n)$ is always contained in $\mathrm{SL}_q^-(n)$, but is open.

6. The cofactor matrix of a q-orthogonal matrix

We consider Problem 5.6(a). For simplicity, assume $n = 2m + 1$ is odd, i.e., we are in Case B. (The other cases are similar.) Proposition 5.1(b) holds for other orderings, for instance relative to the ordering

$$v_1, v_n, v_2, v_{n-1}, \ldots, v_m, v_{m+2}, v_{m+1}.$$

In this section, we use this ordering, and all ordered products of v_i are taken with respect to it.

The n-th component of $\Lambda_q^+(V)$ is spanned by $\bar{v} = v_1 \cdots v_{m+1}$ and the $(n-1)$-th component by

$$w_i = v_1 \cdots \hat{v}_i \cdots v_{m+1}, \qquad 1 \leqq i \leqq n.$$

Lemma 6.1. *In the algebra $\Lambda_q^+(V)$, we have*

$$v_i w_j = \delta_{ij} a_j \bar{v}, \qquad w_i v_j = \delta_{ij} b_j \bar{v}$$

where

$$a_i = \begin{cases} 1, \\ -1, \end{cases} \qquad b_i = \begin{cases} q^{2t} & \textit{if } i \leqq m+1, \\ -q^{2t} & \textit{if } i > m+1. \end{cases}$$

Proof. Exercise left to the reader. (Use 5.1(a).)

Let $J = (W_q^+, W_{q^{1-n}})$ be the defining ideal for $\Lambda_q^+(V)$. For $0 \leqq r \leqq n$, consider the r-th component J_r, together with $J_r^\perp$ in $V^{(r)}$ relative to the canonical inner product (§3). (Note that $\tilde{M}_q^+(n) = \mathcal{M}_n/(\phi(J_2 \otimes J_2^\perp), \phi(J_2^\perp \otimes J_2))$.) We have

$$V^{(n)} = J_n \oplus k\bar{\bar{v}}, \ V^{(n-1)} = J_{n-1} \oplus k\{\bar{w}_i, 1 \leqq i \leqq n\},$$

where $\bar{\bar{v}} = v_1 \otimes \cdots \otimes v_{m+1}$ and $\bar{w}_i = v_1 \otimes \cdots \hat{v}_i \cdots \otimes v_{m+1}$. It follows that there is a base u for $J_n^\perp$ (resp. $u_1, \ldots, u_n$ for $J_{n-1}^\perp$) such that $\langle u, \bar{\bar{v}} \rangle = 1$ (resp. $\langle u_i, \bar{w}_j \rangle = \delta_{ij}$).

Corollary 6.2. *With the notation of 6.1, we have*

$$u = \sum_i a_i v_i \otimes u_i = \sum_i b_i u_i \otimes v_i.$$

PROOF. Since $J_n = V \otimes J_{n-1} + J_{n-1} \otimes V$, we have

$$J_n^\perp = (V \otimes J_{n-1}^\perp) \cap (J_{n-1}^\perp \otimes V).$$

If we write $u = \Sigma_{i,j} \alpha_{ij} v_i \otimes u_j$, we have $\alpha_{ij} = \langle v_i \otimes \bar{w}_j, u \rangle$. On the other hand, Lemma 6.1 means $v_i \otimes \bar{w}_j - \delta_{ij} a_j \bar{v} \in J_n$, whence $\langle v_i \otimes \bar{w}_j, u \rangle = \delta_{ij} a_j$. Similarly we have the second equality. Q.E.D.

LEMMA 6.3. $\langle u_i, u_j \rangle = 0$ *unless* $i = j$.

PROOF. We give a $\mathbf{Z}^m \times \mathbf{Z}/(2)$-gradation on $T(V)$ by imposing

$$\left.\begin{array}{l} \deg v_i = (0, \ldots, \overset{i}{1}, \ldots, \overset{m+1}{0}) \\ \deg v_{i'} = (0, \ldots, -1, \ldots, 0) \end{array}\right\} \quad \text{if } i < i',$$

$$\deg v_{m+1} = (0, \ldots\ldots\ldots, 1).$$

One sees (5.1(a)) J_r and $J_r^\perp$ are graded subspaces of $V^{(r)}$ for all r. It follows that u_i is homogeneous with the same degree as w_i (or $\bar{w}_i$). They are orthogonal to each other, since they have distinct degrees. Q.E.D.

We use the induced comodule structure

$$\rho : V^{(r)} \to V^{(r)} \otimes \tilde{M}_q^+(n).$$

Since J_r is a subcomodule, it follows from Lemma 3.5 that $J_r^\perp$ is also a subcomodule. Let $g = |T|_q$ be the q-determinant of orthogonal type. Since we have $\rho(\bar{v}) = \bar{v} \otimes g$ by definition, it follows that $g =$ (the image of) $\phi(u \otimes \bar{v})$, yielding

$$\rho(u) = u \otimes \theta(g). \tag{6.4}$$

If we put $p_{ij} =$ (the image of) $\phi(\bar{w}_i \otimes u_j)$, we have

$$\rho(u_j) = \sum_i u_i \otimes p_{ij}. \tag{6.5}$$

With the notation of 6.1, let

$$A = \operatorname{diag}(a_1, \ldots, a_n), \qquad B = \operatorname{diag}(b_1, \ldots, b_n)$$

be the diagonal matrices. It follows from 6.2, 6.4, 6.5 that we have

$$A\theta(g) = TAP^t \quad \text{and} \quad B\theta(g) = PBT^t$$

where $T = (t_{ij})$ the canonical generating matrix, and $P = (p_{ij})$. Especially we have

$$Bg = \theta(P)BT.$$

Hence Problem 5.6(a) will be affirmative if the following conjecture is true:

CONJECTURE 6.6. (a) $\theta(g) = g$.
(b) $B^{-1}\theta(P)B = AP^{t}A^{-1}$.

If this is the case, we have $T\tilde{T} = gI = \tilde{T}T$ with $\tilde{T} = AP^{t}A^{-1}$. Note that (b) is equivalent to

$$\phi(q^{2\bar{j}}u_j \otimes \bar{w}_i) = \phi(\bar{w}_j \otimes q^{2\bar{\imath}}u_i), \qquad 1 \leqq i, j \leqq n \tag{6.7}$$

(equality in $\tilde{M}_q^+(n)$), since $a_i b_i = q^{2\bar{\imath}}$.

LEMMA 6.8. *If $\langle u, u \rangle \neq 0$, we have $\theta(g) = g$.*

PROOF. We have $\langle u, u \rangle g = \phi(\langle u, u \rangle u \otimes \bar{v}) = \phi(u \otimes u)$ in $\tilde{M}_q^+(n)$, since $u - \langle u, u \rangle \bar{v} \in J_n^{\perp}$ $(= ku^{\perp})$. Hence g is θ-invariant. Q.E.D.

PROPOSITION 6.9. *If $q^{2\bar{\imath}}\langle u_i, u_i \rangle$ are a non-zero constant c which does not depend on i $(1 \leqq i \leqq n)$, then $B^{-1}\theta(P)B = AP^{t}A^{-1}$.*

PROOF. The equality (6.7) will follow if we show

$$q^{2\bar{j}}u_j \otimes \bar{w}_i - \bar{w}_j \otimes q^{2\bar{\imath}}u_i \in J_{n-1} \otimes J_{n-1}^{\perp} + J_{n-1}^{\perp} \otimes J_{n-1} \tag{6.10}$$

since J_{n-1} and $J_{n-1}^{\perp}$ are conormalized by $\tilde{M}_q^+(n)$. The assumption, together with Lemma 6.3, tells that $V^{(n-1)} = J_{n-1} \oplus J_{n-1}^{\perp}$. Since the left side of (6.10) is orthogonal to $J_{n-1} \otimes J_{n-1}$, it is enough to claim it is also orthogonal to $J_{n-1}^{\perp} \otimes J_{n-1}^{\perp}$. If we take $1 \leqq k, l \leqq n$, we have

$$\begin{aligned} &\langle q^{2\bar{j}}u_j \otimes \bar{w}_i - \bar{w}_j \otimes q^{2\bar{\imath}}u_i, u_l \otimes u_k \rangle \\ &= (q^{2\bar{j}}\langle u_j, u_j \rangle - q^{2\bar{\imath}}\langle u_i, u_i \rangle)\delta_{jl}\delta_{ik} \\ &= 0 \end{aligned}$$

by assumption. Q.E.D.

Assume $q^{2\bar{\imath}}\langle u_i, u_i \rangle$ are a non-zero constant c. It follows from 6.2 that

$$\langle u, u \rangle = \sum_i \langle u_i, u_i \rangle = c \sum_i q^{2\bar{\imath}} = c(q^{n-1} - q^{-1})(1 + q^{2-n})(q - q^{-1})^{-1}.$$

Hence $\theta(g) = g$ by 6.8, unless q is a root of 1.

It is likely that the assumption of 6.9 *is valid, hence Problem* 5.6(a) *is affirmative, unless q is a root of* 1. We can evaluate $q^{2t}\langle u_i, u_i\rangle$ if n is small, but at present it is open for general n.

EXAMPLES 6.11. (a) $n = 3$. We have

$$u_1 = v_3 \otimes v_2 - q v_2 \otimes v_3,$$

$$u_2 = v_1 \otimes v_3 - v_3 \otimes v_1 + (q^{-1/2} - q^{1/2}) v_2 \otimes v_2,$$

$$u_3 = v_1 \otimes v_2 - q^{-1} v_2 \otimes v_1.$$

Hence the assumption of 6.9 is true with $c = q + q^{-1}$. The cofactor matrix $\tilde{T}$ will be

$$\begin{pmatrix} t_{33}t_{22} - qt_{32}t_{23} & t_{13}t_{32} - qt_{12}t_{33} & -t_{13}t_{22} + qt_{12}t_{23} \\ -t_{21}t_{33} + q^{-1}t_{31}t_{23} & t_{11}t_{33} - t_{13}t_{31} - (q^{1/2} - q^{-1/2})t_{12}t_{32} & t_{21}t_{13} - qt_{11}t_{23} \\ -t_{31}t_{22} + q^{-1}t_{32}t_{21} & -t_{11}t_{32} + q^{-1}t_{12}t_{31} & t_{11}t_{22} - q^{-1}t_{12}t_{21} \end{pmatrix}.$$

(b) $n = 5$. Similarly, the assumption of 6.9 is valid with

$$c = (q + q^{-1})^2(q^2 + q^{-2})(q + q^{-1} + 1).$$

Thus Problem 5.6(a) is affirmative for $n = 3, 5$ unless q is a root of 1. When $n = 3$, we can write down explicitly the defining relations for $\tilde{M}_q^+(3)$ [T2, §4]. The result tells that 5.6(b) is also true in this case. It is open for general n.

The matric bialgebra $M_q^+(n)$, which is a quotient of $\tilde{M}_q^+(n)$, has two natural group-like elements γ of degree 2 and $\bar{g}$, the image of g, of degree n. We can naturally expect

$$\gamma^n = \bar{g}^2. \tag{6.12}$$

This means in particular the q-determinant of a q-orthogonal matrix is a square root of 1.

We claim briefly the equality (6.12) is true with some restriction. With the canonical $M_q^+(n)$-comodule structure on $T(V)$, we have $\rho(\bar{v}) = \bar{v} \otimes \bar{g}$ and $\rho(z) = z \otimes \gamma$, with $z = \Sigma_i q^{-t} v_i \otimes v_{i'}$. The map $V^{(2)} \to V^{(4)}$, $x \otimes y \mapsto x \otimes z \otimes y$ is a semi-comodule map relative to the multiplication by γ, which is a coalgebra endomorphism of $M_q^+(n)$. If $z^{[2]}$ denotes the image of z, it follows that

$$\rho(z^{[2]}) = z^{[2]} \otimes \gamma^2.$$

More generally, if $z^{[r]}$ denotes the image of z under the r-times iteration $V^{(2)} \rightarrow V^{(2r)}$, we will have

$$\rho(z^{[r]}) = z^{[r]} \otimes \gamma^r.$$

Hence, if the image of $z^{[n]}$ in $\Lambda_q^+(V) \otimes \Lambda_q^+(V)$ is $d\bar{v} \otimes \bar{v}$ with $d \neq 0$ in k, we can conclude $\gamma^n = \bar{g}^2$.

EXAMPLE 6.13. If $n = 3$, $d = (q + q^{-1})(q + q^{-1} + 1)$. We have (6.12) unless $q^2 = 1$ or $q^3 = 1$.

In characteristic zero, it is easy to see $d \neq 0$ if q is transcendental over $\mathbf{Q}$ (consider modulo $q - 1$).

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